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# Spin-dependent electron-impurity scattering in two-dimensional electron systems

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We present a theoretical study of elastic spin-dependent electron scattering caused by a charged impurity in the vicinity of a two-dimensional electron gas. We find that the symmetry properties of the spin-dependent differential scattering cross section are different for an impurity located in the plane of the electron gas and for one at a finite distance from the plane. We show that in the latter case asymmetric ("skew") scattering can arise if the polarization of the incident electron has a finite projection on the plane spanned by the normal vector of the two-dimensional electron gas and the initial propagation direction. In specially prepared samples this scattering mechanism may give rise to a Hall-type effect in the presence of an *in-plane* magnetic field.

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In quantum-scattering theory one of the central quantities is the differential scattering cross section (DSCS). In the so-called *S*-matrix formalism it is possible to derive the symmetry properties of the DSCS if the symmetries of the Hamiltonian—the operators commuting with the Hamiltonian—are known.[1,2] A particular example was studied by Huang *et al.*[3] They considered the elastic scattering of two-dimensional (2D) electrons off a charged impurity sitting in the middle of a semiconductor quantum well, taking into account the spin-orbit coupling (SOC) created around the impurity. They have found that despite the cylindrical symmetry of the electrostatic potential created by the impurity, the DSCS can be asymmetric with respect to the forward-scattering direction, and its antisymmetric component is proportional to the out-of-plane component of the polarization vector of the incoming electron. This effect is called asymmetric or skew scattering and is reminiscent of the so-called Mott skew scattering in three dimensions.[4–6]

This special scattering behavior caused by the SOC around the impurity can have a directly measurable consequence on the transport of spin-polarized carriers, the so-called anomalous Hall effect (AHE). The AHE in bulk metals has been in the focus of experimental and theoretical researches for many decades,[7–10] and recent advances in the field of magnetic semiconductors have increased the activity within this area even further.[11–15]

Recently Cumings *et al.*[16] observed the AHE in a paramagnetic two-dimensional electron gas (2DEG) created in a semiconductor quantum well, and its appearance was attributed to the asymmetric electron-impurity scattering. In this experiment an out-of-plane magnetic field was applied, resulting in the Lorentz force acting on the moving electrons and a finite spin polarization of the carriers via the Zeeman effect. The Lorentz force alone would result in the well-known normal Hall resistivity, but the simultaneous presence of the finite spin polarization and the skew scattering process gives rise to an additional anomalous Hall component.

The experiment of Cumings *et al.*[16] has confirmed that the electron-impurity scattering can contribute significantly to the Hall resistivity in paramagnetic 2DEGs. Motivated by this fact, in this work we extend the problem considered by Huang *et al.*[3] and study the spin-dependent electron-impurity scattering process in a 2DEG where the impurity might be located at any finite distance from the plane of the 2DEG. We focus our attention to the symmetry properties of the DSCS describing individual scattering processes, and we show that the properties of the DSCS are fundamentally different when the impurity is located in the middle of the quantum well and when it is at a finite distance from that. In the former case skew scattering arises only if the polarization vector of the incident electron has a finite out-of-plane component. In contrast, we find that in the latter case skew scattering happens provided that the polarization vector of the incident electron has a finite projection on the plane spanned by the normal vector of the plane of the 2DEG and the initial propagation direction. In the first part of this Rapid Communication we summarize the rigorous quantum-mechanical derivation of this result. To provide a simple physical picture of this scattering process we also present a classical analysis of the electron dynamics affected by the impurity. Finally, we discuss a possible experimental setup where the special feature of the considered scattering process could give rise to a Hall-type effect in the presence of an *in-plane* magnetic field.

We consider a 2DEG in the *x*-*y* plane created in a symmetric quantum well and a charged pointlike impurity in the position $\boldsymbol{r}_0=(0,0,z_0)$. The setup is shown in Fig. 1. Assuming that the SOC strength $\lambda$ is energy independent and has the same value in the quantum well and barrier materials, the contribution of the impurity to the Hamiltonian is[17]

$$H_i = V_i(\boldsymbol{r}) + \frac{\lambda e}{\hbar}\boldsymbol{E}_i(\boldsymbol{r})\cdot(\boldsymbol{\sigma}\times\boldsymbol{p}). \quad (1)$$

Here $\boldsymbol{r}$ is three-dimensional coordinate vector of the electron, $V_i$ is the electrostatic potential created by the impurity, $\boldsymbol{E}_i=\boldsymbol{\nabla}V_i/e$ is the electrostatic field created by the impurity,

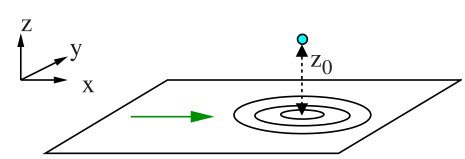


FIG. 1. (Color online) The electron in a 2DEG (represented by the green/light gray arrow) approaches the impurity located in the position $\boldsymbol{r}_0=(0,0,z_0)$. Circular lines represent the equipotentials of the impurity potential.

 

and $\boldsymbol{\sigma}$ is the vector of Pauli matrices representing electron spin.

We start with the quantum-mechanical analysis of the electron dynamics in this system. For simplicity the 2DEG is treated as ideal in the sense that electrons are confined to the plane. Under this assumption, the system can be modeled by the following effective two-dimensional Hamiltonian:

$$H_{2D} = \frac{p_x^2 + p_y^2}{2m^*} + \bar{V}_i + \frac{\lambda e}{\hbar}\sigma_z(\bar{E}_{i,y}p_x - \bar{E}_{i,x}p_y) + \frac{\lambda e}{2\hbar}[\sigma_x\{p_y, \bar{E}_{i,z}\} - \sigma_y\{p_x, \bar{E}_{i,z}\}], \quad (2)$$

where for any $f \in \{V_i, E_{i,x}, E_{i,y}, E_{i,z}\}$ we have defined the notation $\bar{f}(x,y) = f(x,y,0)$ and $\{.,.\}$ denotes the anticommutator. This form of the Hamiltonian can be derived in a rigorous way using the standard dimension reduction technique used by Huang *et al.*,[3] assuming that the energy dependence of the effective mass $m^*$ and SOC strength $\lambda$ is negligible. Note that the cylindrical symmetry of the electrostatic potential $\bar{V}_i$ and field $\bar{\boldsymbol{E}}_i$ created by the impurity is retained even if screening effects are incorporated into $V_i$.

Having the effective 2D Hamiltonian $H_{2D}$ in hand, now it is possible to study the scattering of electrons on the spin-dependent impurity potential. In the absence of the impurity, $H_{2D}$ has plane-wave eigenfunctions. We will consider the scattering of the electron plane wave,

$$\phi_\gamma(\rho, \varphi) = e^{ik\rho\cos\varphi}\gamma, \quad (3)$$

which has energy $E = \hbar^2 k^2 / 2m^*$ and propagates along the $x$ axis. Here $\rho$ and $\varphi$ denotes the standard planar polar coordinates and $\gamma$ is a normalized two-component complex vector describing the spin state of the plane wave. We denote the polarization vector of the incident electron by $\boldsymbol{P}_0$, which is a three-dimensional real unit vector and is related to the spinor $\gamma$ by the expression $\boldsymbol{P}_0 = \gamma^\dagger \boldsymbol{\sigma} \gamma$. Here $\dagger$ denotes the combination of complex conjugation and transposition.

It has been shown[18] that in two-dimensional spin-dependent electron-scattering problems the DSCS can be expressed as the function of the scattering angle $\varphi$ and the polarization vector of the incident electron $\boldsymbol{P}_0$ in the following form:

$$\sigma_{\text{diff}}(\varphi, \boldsymbol{P}_0) = c(\varphi) + \boldsymbol{v}(\varphi) \cdot \boldsymbol{P}_0. \quad (4)$$

Here the dot represents scalar product and the function $c(\varphi)$ and the vector-valued function $\boldsymbol{v}(\varphi) = [v_1(\varphi), v_2(\varphi), v_3(\varphi)]$ are related to $S$ matrix and the scattering amplitude.

It can be shown straightforwardly that the Hamiltonian in Eq. (2) has three important symmetries: $H_{2D}$ commutes with the out-of-plane component of the total angular-momentum operator ($J_z = -i\hbar\partial_\varphi + \hbar\sigma_z/2$), with time reversal ($T = i\sigma_y C$ where $C$ is the complex conjugation) and with a special combined symmetry of real-space reflection and spin rotation ($\sigma_y P_x$, where $P_x$ is the spatial reflection with respect to the $x$ axis). However, if $z_0 = 0$, i.e., the impurity is located in the plane of the quantum well, then an additional symmetry of the Hamiltonian can be found. Namely, in this case the out-of-plane component $\bar{E}_{i,z}$ of the electric field created by the impurity vanishes identically and so does the last term in the Hamiltonian in Eq. (2). It means that in this special $z_0 = 0$ case $\sigma_z$ also commutes with $H_{2D}$, and as a consequence of that the symmetry properties of the DSCS are different in the cases $z_0 = 0$ and $z_0 \neq 0$, as it will be shown below.

TABLE I. Symmetry properties of the quantities determining the differential scattering cross section in Eq. (4). S (A) denotes the quantities which are even (odd) function of the scattering angle $\varphi$. 0 denotes the quantities which vanish identically as a result of the symmetries of the system.

| Case | $c$ | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|---|
| Exact $z_0 = 0$ | S | 0 | 0 | A |
| Exact $z_0 \neq 0$ | S | A | S | A |
| Born $z_0 = 0$ | S | 0 | 0 | 0 |
| Born $z_0 \neq 0$ | S | A | S | 0 |

Starting from these symmetry properties of the Hamiltonian $H_{2D}$ in Eq. (2) and following the method using the $S$-matrix formalism outlined in Ref. 2, we were able to derive the symmetry properties of the functions $c$ and $\boldsymbol{v}$ appearing in the formula [Eq. (4)] of the DSCS. Our findings are summarized in the first and second lines of Table I. In the $z_0 = 0$ case, in correspondence with previous results,[3] we have found that skew scattering can arise only if the out-of-plane component of the polarization vector of the incident electron is finite. This can be deduced using the first line of Table I and Eq. (4). On the other hand, a fundamentally different behavior is found in the $z_0 \neq 0$ case when the impurity is lifted out from the 2DEG plane. In this case, skew scattering is forbidden only if the initial polarization vector is aligned with the $y$ axis, and the DSCS can become asymmetric if the initial polarization vector has a finite component in the $x$-$z$ plane [see the second line of Table I and Eq. (4)]. In general, in the $z_0 \neq 0$ case skew scattering happens provided that the initial polarization vector $\boldsymbol{P}_0$ has a finite projection on the plane spanned by the normal vector of the 2DEG and the initial propagation direction.

Usually, the symmetry properties of the DSCS calculated in the first Born approximation (FBA) are more restrictive than those of the exact DSCS. A particular example is the $z_0 = 0$ case considered in Ref. 3, where it has been shown that the quantity $\boldsymbol{v}(\varphi)$ in Eq. (4) characterizing the spin-dependent component of the DSCS vanishes if the scattering problem is treated in the FBA. (Compare first and third lines of Table I.) We have applied a similar analysis in the $z_0 \neq 0$ case and found that in this case $\boldsymbol{v}(\varphi)$ can be finite in the FBA. Moreover, we derived its symmetry properties as well and summarized the results in the fourth line of Table I. To give a quantitative estimate of the skewness of the DSCS for an electron having its initial polarization vector aligned with its propagation direction, we calculated the quantity $v_1(\varphi)/c(\varphi)$ in the FBA for a screened impurity with charge $Ze$. The screening has been taken into account by means of the Thomas-Fermi model.[19] The leading-order result in the dimensionless parameter $\lambda k^2$ is

$$\frac{v_1(\varphi)}{c(\varphi)} = \mathrm{sgn}(z_0)\lambda k^2 \sin\varphi\left(2\sin\frac{|\varphi|}{2} + \frac{q_0}{k}\right), \quad (5)$$

where $q_0 = m^* e^2/(2\pi\hbar^2\epsilon_0\kappa)$ is the Thomas-Fermi wave number, $\epsilon_0$ is the vacuum permittivity, $\kappa$ is the dimensionless relative permittivity of the material, sgn is the sign function, and $\varphi \in [-\pi, \pi]$. For the derivation of Eq. (5) we assumed that $q_0/k \lesssim 1$. In typical 2DEG samples the relations $\lambda k_F^2 \ll 1$ and $q_0/k_F \lesssim 1$ usually hold for the Fermi wave number $k_F$, and therefore the scattering of mobile electrons is well described by the formula [Eq. (5)]. For example, in an InAs quantum well with electron sheet density $n_s = 10^{12}$ cm$^{-2}$, effective mass $m^* = 0.023 m_0$, relative permittivity $\kappa = 15$, and SOC strength[6] $\lambda \approx 120$ Å$^2$, we get $\lambda k_F^2 \approx 0.075$ and $q_0/k_F \approx 0.25$. Note that the quantity $v_1/c$ can be regarded as a generalization of the Sherman function[5] used for characterizing the skewness of the spin-dependent DSCS of three-dimensional Mott scattering.

Now we present a simple classical interpretation of the derived symmetry properties characterizing the considered spin-dependent scattering process. In the following we use the quantum Hamiltonian in Eq. (1) to derive equations of motion for the observables and otherwise we treat $\boldsymbol{r}$, $\boldsymbol{p}$, and $\boldsymbol{\sigma}$ as strictly classical quantities. Specially, instead of $\boldsymbol{\sigma}$ we will use the three-dimensional unit vector $\boldsymbol{P}$.

Consider a pointlike particle in the 2DEG approaching the scattering center along the $x$ axis with impact parameter $b$ and spin-polarization vector $\boldsymbol{P}_0$. For simplicity we assume that the two-dimensional classical dynamics of the scattered electron is determined mainly by the electrostatic potential $V_i$, and the SOC plays the role of a weak perturbation. The trajectory of the motion affected only by $V_i$ (the "unperturbed" trajectory) is given by $\boldsymbol{r}(t,b) = [x(t,b), y(t,b), 0]$, the position vector of the particle with impact parameter $b$ at time $t$. We assume that at the moment $t=0$ the particle is approaching the scattering center but still out of the range of the potential created by the impurity. Trivially

$$x(t,b) = x(t,-b), \quad (6a)$$

$$y(t,b) = -y(t,-b), \quad (6b)$$

i.e., the unperturbed trajectories corresponding to impact parameters $b$ and $-b$ are related by a reflection with respect to the $x$ axis. During the motion the SOC [the second term in Eq. (1)] acts as an effective inhomogeneous magnetic field felt by the electron spin: $\boldsymbol{P}\cdot\boldsymbol{B}_{\mathrm{eff}}(\boldsymbol{r},\dot{\boldsymbol{r}})$, where

$$\boldsymbol{B}_{\mathrm{eff}}(\boldsymbol{r},\dot{\boldsymbol{r}}) = -\frac{\lambda e m^*}{\hbar}[\boldsymbol{E}(\boldsymbol{r}) \times \dot{\boldsymbol{r}}]. \quad (7)$$

The dot ($\dot{\boldsymbol{r}}$) denotes time derivative, and we used $\dot{\boldsymbol{r}} = \boldsymbol{p}/m^*$. With respect to the spin and orbital dynamics, there are two important consequences of the presence of this inhomogeneous effective magnetic field. First, the spin of the moving particle will precess around the effective magnetic field. Second, the inhomogeneity of the effective magnetic field gives rise to a Stern-Gerlach-type force which deflects the particle from its unperturbed trajectory.

For a given impact parameter $b$ and unperturbed trajectory $\boldsymbol{r}(t,b)$, the equation of motion for the spin of the moving electron is[1]

$$\dot{\boldsymbol{P}}(t,b) = \frac{1}{\hbar}\boldsymbol{P}(t,b) \times \boldsymbol{B}_{\mathrm{eff}}[\boldsymbol{r}(t,b),\dot{\boldsymbol{r}}(t,b)], \quad (8)$$

similarly to the well-known Bloch equations.[20] If the spin of the incident particle $\boldsymbol{P}_0 \equiv \boldsymbol{P}(t=0,b)$ is given, and with this initial value condition the solution of Eq. (8) is known, then the Stern-Gerlach force[1] can be expressed as the gradient of the local SOC energy,

$$\boldsymbol{F}[\boldsymbol{r}(t,b),\boldsymbol{P}(t,b)] = -\nabla\{\boldsymbol{P}(t,b)\cdot\boldsymbol{B}_{\mathrm{eff}}[\boldsymbol{r}(t,b),\dot{\boldsymbol{r}}(t,b)]\}. \quad (9)$$

The presence of this force is due to the presence of SOC close to the scattering center, and it deflects the particles from their original, unperturbed trajectory.

We claim that if the spin of the incident electron has a finite projection on the $x$-$z$ plane then the presence of the SOC destroys the reflection symmetry of the motion with respect to the $x$ axis and therefore gives rise to an asymmetry in the DSCS as well. This is a consequence of the fact that if the polarization vector lies in the $x$-$z$ plane then the components of the Stern-Gerlach force fulfill the relations

$$F_1[\boldsymbol{r}(t,b),\dot{\boldsymbol{r}}(t,b)] = -F_1[\boldsymbol{r}(t,-b),\dot{\boldsymbol{r}}(t,-b)], \quad (10a)$$

$$F_2[\boldsymbol{r}(t,b),\dot{\boldsymbol{r}}(t,b)] = F_2[\boldsymbol{r}(t,-b),\dot{\boldsymbol{r}}(t,-b)]. \quad (10b)$$

(Here and henceforth the vector subscripts 1, 2, and 3 are equivalent to $x$, $y$, and $z$, respectively.) We sketch the steps of the proof in the following. The definition of $\boldsymbol{B}_{\mathrm{eff}}$ in Eq. (7), the properties in Eqs. (6), and the cylindrical symmetry of $\boldsymbol{E}(\boldsymbol{r})$ imply symmetry relations of the components of $\boldsymbol{B}_{\mathrm{eff}}$,

$$\{\boldsymbol{B}_{\mathrm{eff}}[\boldsymbol{r}(t,b),\dot{\boldsymbol{r}}(t,b)]\}_k = (-1)^k\{\boldsymbol{B}_{\mathrm{eff}}[\boldsymbol{r}(t,-b),\dot{\boldsymbol{r}}(t,-b)]\}_k, \quad (11)$$

where $k=1,2,3$. The Picard-Lindelof[21] solution of Eq. (8) is

$$\boldsymbol{P}(t,b) = \boldsymbol{P}_0 + \int_0^t dt' \frac{\boldsymbol{B}_{\mathrm{eff}}(t',b)}{-\hbar} \times \boldsymbol{P}_0 + \int_0^t dt' \int_0^{t'} dt'' \frac{\boldsymbol{B}_{\mathrm{eff}}(t',b)}{-\hbar} \times \frac{\boldsymbol{B}_{\mathrm{eff}}(t'',b)}{-\hbar} \times \boldsymbol{P}_0 + \cdots, \quad (12)$$

where we used the notation $\boldsymbol{B}_{\mathrm{eff}}(t,b) \equiv \boldsymbol{B}_{\mathrm{eff}}[\boldsymbol{r}(t,b),\dot{\boldsymbol{r}}(t,b)]$ for brevity. Using Eqs. (11) and (12) and assuming an initial spin $\boldsymbol{P}_0 \perp (0,1,0)$, it is straightforward to prove that

$$P_k(t,b) = -(-1)^k P_k(t,-b). \quad (13)$$

Finally, substituting Eqs. (11) and (13) into Eq. (9) results in Eqs. (10). A similar analysis shows that if the spin of the incident electron is aligned with the $y$ axis then the reflection symmetry between the trajectories corresponding to $b$ and $-b$ is retained in the presence of SOC, and therefore this is the only case when skew scattering does not take place. As a generalization of these results, we formulate the central theorem of this classical analysis as follows: if the initial polarization vector has a finite projection on the plane spanned by the normal of the 2DEG and the initial propagation direction, then the DSCS becomes *asymmetric* with respect to forward-scattering direction. Note that this conclusion is equivalent to the results of the rigorous quantum-mechanical symmetry analysis presented above.

By repeating the preceding analysis for the case when the impurity is located in the plane of the 2DEG ($z_0=0$), and using the fact that in this case the out-of-plane component of the electric field vanishes identically, it can be shown that skew scattering may occur only if the out-of-plane component of the initial polarization vector of the electron is finite. This result is in agreement with the quantum-mechanical result summarized in the first line of Table I.

So far we have discussed the symmetry properties of the DSCS corresponding to individual scattering events. Here we argue that in specially prepared disordered samples this peculiar scattering mechanism may give rise to an experimentally observable effect similar to the skew scattering induced AHE in spin-polarized systems. The setup is shown in Fig. 2. The sample for the proposed experiment should contain a symmetric quantum well with an additional delta-doped impurity layer at a finite distance from the quantum well. For clarity in Fig. 2 we show only a single impurity. We also assume that the 2DEG is fully or partially spin-polarized by a static homogeneous in-plane magnetic field $\boldsymbol{B}$. If a finite direct current parallel to the magnetic field is flowing through this sample, then the skew scattering mechanism will result in a finite Hall signal despite the fact that the magnetic field has no out-of-plane component: skew scattering means that the electrons drifting in the direction of the driving electric field are preferably scattered by impurities to, say, the left (right) if their spin is parallel (antiparallel) to the magnetic field. Therefore in the polarized 2DEG more electrons will pile up at the left edge of the sample than at the other edge, giving rise to a finite transversal bias between the two edges. Though quantitative estimates regarding the magnitude of this effect are not presented here, the fact that $v_1$ is finite in the FBA while $v_3$ vanishes suggests that the predicted effect should be at least comparable with the anomalous Hall effect[16] measured in the presence of an out-of-plane polarization.

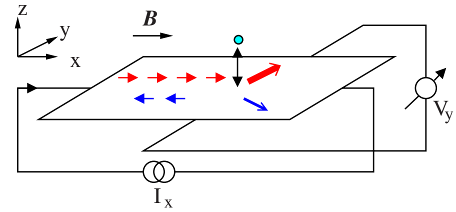


FIG. 2. (Color online) Proposed measurement setup for experimental investigation of the Hall-type effect if an impurity plane is present in the sample at a finite distance from the 2DEG plane. The short horizontal red/gray (blue/dark gray) arrows in the figure represent spins parallel (antiparallel) to the polarizing magnetic field $\boldsymbol{B}$.

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